ATS-16621

IFSM-73-44

INSTITUTE OF FRACTURE AND SOLID MECHANICS

LEHIGH UNIVERSITY

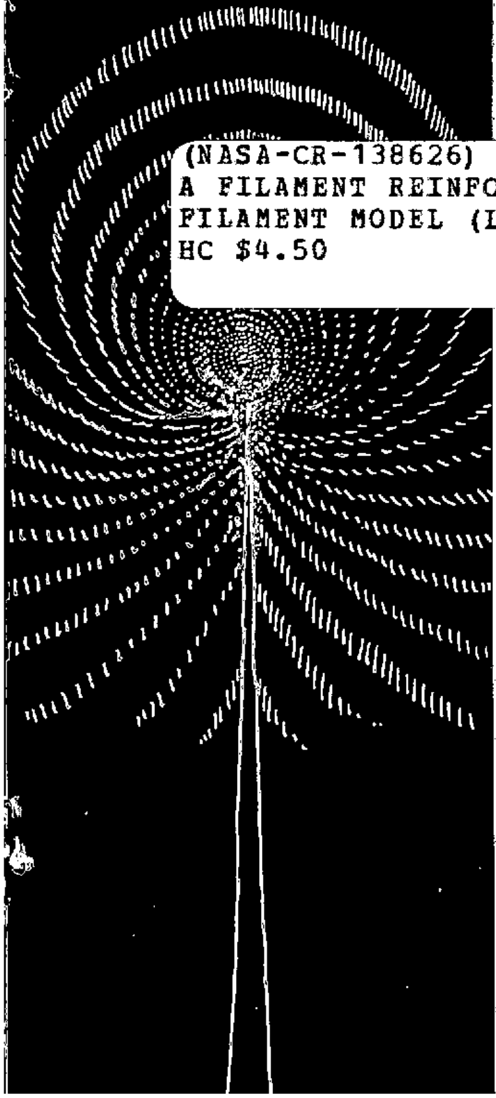
A PENNY-SHAPED CRACK IN A FILAMENT-REINFORCED MATRIX
I. THE FILAMENT MODEL

BY

F. ERDOGAN

AND

A. H. PACELLA



TECHNICAL REPORT NASA TR-73-13

JULY 1973

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

GRANT NGR-39-007-011

A PENNY-SHAPED CRACK IN A FILAMENT-REINFORCED MATRIX*
I. THE FILAMENT MODEL

by

F. Erdogan and A. H. Pacella**
Lehigh University, Bethlehem, Pa.

Abstract. This study deals with the elastostatic problem of a penny-shaped crack in an elastic matrix which is reinforced by filaments or fibers perpendicular to the plane of the crack. An elastic filament model is developed in the first paper. The second paper considers the application of the model to the penny-shaped crack problem in which the filaments of finite length are symmetrically distributed around the crack. The reinforcement problem for the cracked matrix with elastic fibers of different diameter, modulus, and relative location is considered in the third paper. Since the primary interest is in the application of the results to studies relating to the fracture of fiber or filament-reinforced composites and reinforced concrete, the main emphasis of the study will be on the evaluation of the stress intensity factor along the periphery of the crack, the stresses in the filaments or fibers, and the interface shear between the matrix and the filaments or fibers.

1. INTRODUCTION

The primary objective of this series of papers is to develop a technique by which, with a reasonable amount of computational effort, one may obtain the solution of the three-dimensional elasticity problem for a matrix containing a penny-shaped crack and reinforced by elastic filaments or fibers perpendicular to the plane of the crack. Basically, the problem is one of

---

* This work was supported by The National Science Foundation under the Grant GK-11977 and by the National Aeronautics and Space Administration under the Grant NGR-39-007-011.

**Present Address: Bettis Atomic Power Lab., Westinghouse Electric Corp., West Mifflin, Pa.

interaction between a macroscopic crack and filaments or fibers in a composite medium. The problem finds its practical applications in the fracture studies of fiber or filament reinforced composites and in reinforced concrete. Even though the existence of such a crack has an effect on the vibration characteristics, the stiffness, and other mechanical properties of the material, its main importance lies in the reduction it causes in the fracture resistance of the structure. Hence, our primary attention will be concentrated on the calculation of such quantities as the distribution of the stress intensity factor along the periphery of the crack, the filament-matrix shear stress and the maximum tensile stress in the filaments.

This first paper in the series will be devoted to the development of a model for an elastic filament imbedded into an elastic matrix. The main requirements expected of the model are a sufficiently accurate representation of the filament, and its applicability to the interaction problems involving a cracked elastic continuum with multi-filament reinforcements. For a sparsely reinforced matrix in which the interaction between the perturbed stress fields of the isolated filaments and the crack is negligible, the solution given in [1] for an ellipsoidal inclusion in an infinite matrix may be quite satisfactory provided the filament ends are rounded and there is no excessive concentration of interface shear. However, since the filaments are usually cylindrical with sharp edges and since the technique described in [1] cannot readily be expanded to interaction problems, the ellipsoidal inclusion model of [1] is not suitable for the problem under

consideration. A somewhat more appropriate model for the present purpose would be that described in [2]. The model described in [2] would give sufficiently accurate results for the tensile stress in the filament and for the stiffening effect on the crack. However, its representation of the filament-matrix contact stresses would not be sufficiently accurate. Partly for this reason and partly for reasons of convenience in solving the resulting integral equations, in this paper a somewhat different model will be developed. The technique is based on a direct generalization of the notions discussed in [3] and [4], and will be described in the next section. Some numerical examples will then be given and the results will be compared with those obtained from using the methods of [1] and [2].

2. SOLUTION OF THE GENERAL INCLUSION PROBLEM

Consider the three-dimensional inclusion problem shown in Figure 1. Let the homogeneous, isotropic elastic domains D_k (the inclusions) which are bounded by nonintersecting smooth surfaces S_k, $(k = 1,2,\dots,m)$ be perfectly bonded to the surrounding elastic medium D_o (the matrix). Let the bounding surface S_o of D_o be subjected to surface tractions T_i^{no} (where S_o may be finite or infinite). Let the elastic constants of D_k be μ_k, λ_k, $(k = 0,1,\dots,m)$. The problem may be formulated by writing the field equations for D_k separately with the boundary conditions on S_o and stress and displacement continuity conditions on $S_1,\dots,S_m$. This, however, requires the solution of an elasticity problem for the simple domains $D_1,\dots,D_m$ as well as for the multiply-connected domain D_o. The problem may also be considered as that of a

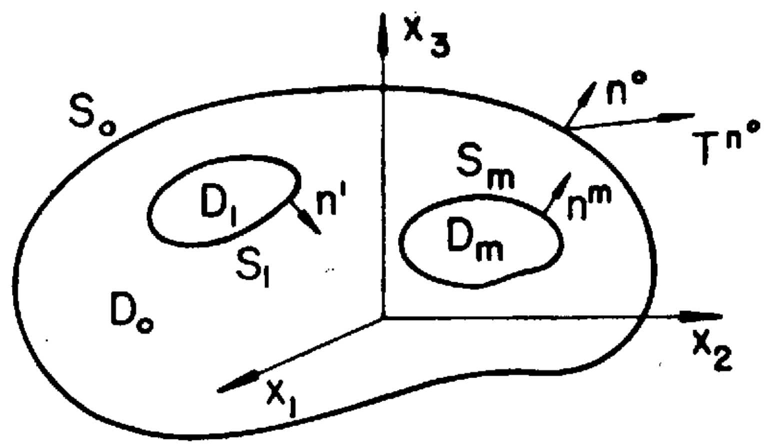


Figure 1. General inclusion geometry.

simply-connected nonhomogeneous domain in which the elastic constants have jump discontinuities along the surfaces $S_1,\dots,S_m$. This formulation requires the solution of a problem in which the field equations have discontinuous coefficients. Aside from the special case discussed in [1], neither one of these solutions is tractable. However, it can be shown that the problem may be reduced to the solution of a system of integral equations provided the Poisson's ratios of the elastic domains $D_0,D_1,\dots,D_m$ are assumed to be equal. For certain geometries these integral equations may be solved numerically without any difficulty.

Let D and S be the union of inclusion domains and their boundaries, respectively, i.e.,

$$D = \sum_1^m D_k \ , \qquad S = \sum_1^m S_k \tag{1}$$

Let u_1,u_2,u_3 be the components of the displacement vector in the nonhomogeneous medium $(D_0 + D + S)$. In the absence of body forces, the elastostatic boundary value problem may be formulated as

$$\mu u_{i,jj} + (\lambda+\mu)u_{j,ji} = 0 \ , \qquad (x_j \in (D+S+D_0)), \tag{2}$$

$$\sigma_{ij}n_j^o = T_i^{n^o} \ , \qquad (x_j \in S_0), \tag{3}$$

$$\sigma_{ij} = \mu(u_{i,j} + u_{j,i}) + \lambda u_{k,k}\delta_{ij} \ , \qquad (i,j=1,2,3), \tag{4}$$

where the discontinuous elastic constants are given by

$$\mu = \mu_r \ , \qquad \lambda = \lambda_r \ , \qquad (x_j \in D_r,\ r=0,1,\dots,m), \tag{5}$$

n_j^o is the outward normal and $T_i^{n^o}$ is the traction vector on S_o. In (2) - (4) as well as in the rest of this paper the usual summation convention is used. Let

$$\mu = \mu_o + \Delta\mu \ , \qquad \lambda = \lambda_o + \Delta\lambda \ ; \tag{6}$$

$$\Delta\mu = \mu_r - \mu_o \ , \quad \Delta\lambda = \lambda_r - \lambda_o \ , \quad (x_j \in D_r, \ r = 1,\ldots,m), \tag{7}$$

where $\Delta\mu$ and $\Delta\lambda$ are assumed to be nonzero. With (6), (2) may be expressed as

$$\mu_o u_{i,jj} + (\lambda_o + \mu_o)u_{j,ji} + [\Delta\mu u_{i,jj} + (\Delta\lambda + \Delta\mu)u_{j,ji}] = 0 \ ,$$
$$(x_j \in (D_o + D + S); \ i = 1,2,3). \tag{8}$$

In (8) the quantity in brackets may be considered as a body force vector which, due to the discontinuous nature of the coefficients λ and μ, is expected to be discontinuous across and at the boundary S. Let $\eta = \eta(x_j)$, $(x_j \in S)$ be the distance along the normal n measured (in outward direction) from the surface S. We may then define

$$\Delta\mu u_{i,jj} + (\Delta\lambda + \Delta\mu)u_{j,ji} = F_i'(x_j) \ , \qquad (x_j \in D + S)$$

$$F_i' = \begin{cases} F_i(x_j) \ , & (x_j \in D), \\ T_i^n(x_j)\delta(\eta - 0) \ , & (x_j \in S), \end{cases} \tag{9.a,b}$$

where, in general, T_i^n is not equal to the boundary value F_i^-. Using (9), (8) may now be expressed as

$$\mu_o u_{i,jj} + (\lambda_o + \mu_o)u_{j,ji} + F_i + T_i^n\delta(\eta - 0) = 0 \ ,$$
$$(x_j \in (D_o + D + S), \ i = 1,2,3). \tag{10}$$

On the other hand if we let $F_i(x_j) = F_i^r$, $(x_j \in D_r)$ and $T_i^n(x_j) = T_i^{n^r}$, $(x_j \in S_r)$, (9) is equivalent to

$$\Delta\mu_r u_{i,jj}^r + (\Delta\lambda_r + \Delta\mu_r)u_{j,ji}^r - F_i^r = 0 \ , \qquad (x_j \in D_r),$$

$$\sigma^r_{ij} n^r_j = - T^{n^r}_i , \qquad (x_j \in S_r),$$

$$\sigma^r_{ij} = \Delta\mu_r (u^r_{i,j} + u^r_{j,i}) + \Delta\lambda u^r_{k,k} \delta_{ij} ,$$

$$(r = 1,\ldots,m;\ i,j = 1,2,3). \qquad (11.a\text{-}c)$$

(10) with (3) and (4), and (11) give the formulation of m+1 elasticity problems for the simply-connected homogeneous domains $(D_o+S+D), D_1,\ldots,D_m$. In addition to displacement components $u_i = u^o_i$ in (D_o+D+S) and u^r_i in the auxiliary inclusions D_r with the elastic constants $\Delta\mu_r$ and $\Delta\lambda_r$, $(r = 1,\ldots,m;\ i = 1,2,3)$, the equations contain the unknown functions F^r_i and $T^{n^r}_i$, $(r = 1,\ldots,m;$ $i = 1,2,3)$. The additional equations to account for these unknowns may be obtained by considering the fact that the displacements on S are continuous and u^r_i given by (11) are identical to u_i given by (9), (8), or (10), namely

$$u^r_i(x_j) = u^o_i(x_j) , \qquad (x_j \in D_r),$$

$$u^r_i(x_j) = u^o_i(x_j) , \qquad (x_j \in S_r),\ (r = 1,\ldots,m;\ i = 1,2,3).$$

$$(12.a,b)$$

Formally, 9m+3 unknown functions u^o_i, u^r_i, F^r_i, $T^{n^r}_i$ may be obtained from 9m+3 equations given by (10), (11.a), and (12) under the boundary conditions (3) (with (4)) and (11.b) (with (11.c)). If the Green's functions for the domains (D_o+D+S) (i.e., the simply-connected region bounded by S_o) and S_r, $(r = 1,\ldots,m)$ are known, this system of equations may easily be replaced by a system of 6m integral equations for the unknown functions F^r_i and $T^{n^r}_i$, $(i = 1,2,3;\ r = 1,\ldots,m)$.

Let us now consider the field equations for the part D_r of

the homogeneous medium (D_0+D+S) in which $u_i(x_j) = u_i^o(x_j)$. This may be obtained from (10) as

$$u^o_{i,jj} + (1 + \frac{\lambda_o}{\mu_o}) u^o_{j,ji} + \frac{1}{\mu_o} F^r_i = 0 ,$$

$$(x_j \in D_r;\ r = 1,\ldots,m;\ i,j = 1,2,3), \qquad (13)$$

subject to the boundary conditions that $\sigma_{ij}n^r_j = \sigma^{o-}_{ij}n^r_j$, ($x_j \in S_r$) where σ^{o-}_{ij} is the limit of the stress component obtained from the solution of (10) as x_j approaches the boundary S_r from inside. From (13), (11.a) and (12.b) it is easily seen that

$$\left(\frac{\lambda_o}{\mu_o} - \frac{\Delta\lambda_r}{\Delta\mu_r} \right) u^o_{j,ji} + \left(\frac{1}{\mu_o} + \frac{1}{\Delta\mu} \right) F^r_i = 0 , \qquad (x_j \in D_r). \qquad (14)$$

From (14) it then follows that

$$F^r_i \equiv 0 ,\ \text{if}\ \nu_r = \nu_o ,\quad (x_j \in D_r;\ r = 1,\ldots,m;\ i = 1,2,3), \qquad (15)$$

where ν_s is the Poisson's ratio of the elastic region D_s, ($s = 0,1,\ldots,m$). Thus, with the assumption that $\nu_r = \nu_o$, ($r = 1,\ldots,m$) the formulation of the problem may be considerably simplified and may be summarized as

$$\mu_o u^o_{i,jj} + (\lambda_o + \mu_o) u_{j,ji} + \sum_{r=1}^{m} T^{n^r}_i \delta(n^r - 0) = 0 ,$$

$$(x_j \in (D_o\text{+D+S})),$$

$$[\mu_o(u^o_{i,j} + u^o_{j,i}) + \lambda_o u^o_{k,k}\delta_{ij}]n^o_j = T^{n^o}_i , \quad (x_j \in S_o), \quad (16.a,b)$$

$$\Delta\mu_r u^r_{i,jj} + (\Delta\lambda_r + \Delta\mu_r) u^r_{j,ji} = 0 , \qquad (x_j \in D_r),$$

$$[\Delta\mu_r(u^r_{i,j} + u^r_{j,i}) + \Delta\lambda_r u^r_{k,k}\delta_{ij}]n^r_j = - T^{n^r}_i ,$$

$$(x_j \in S_r), \qquad (17.a,b)$$

$$u^r_i(x_j) = u^o_i(x_j) , \quad (x_j \in S_r),\ (r = 1,\ldots,m;\ i,j = 1,2,3). \qquad (18)$$

Again, if the Green's functions for the regions $D_1,\ldots,D_m$ for a concentrated stress vector on the boundary and for (D_0+D+S) for a concentrated internal body force are known,(18) directly gives a system of two-dimensional integral equations for the unknown functions $T_i^{n^r}$, $(i = 1,2,3;\ r = 1,\ldots,m)$.

In the study of the mechanics of composite materials an important quantity of interest is the magnitude of the contact stresses on the interfaces $S_1,\ldots,S_m$. Once the layers of body forces $T_i^{n^r}$, $(r = 1,\ldots,m)$ are determined, the contact stresses may easily be obtained from the equilibrium considerations along the boundaries $S_1,\ldots,S_m$. Let $\tau_i = \sigma_{ij}n_j$ be the components of the stress vector on the internal surface S, $(S = \sum_1^m S_r)$ having the normal (n_i), $(i,j = 1,2,3)$. Let τ_i^r be the components of the contact stress vector on the interface S_r having the outward normal n_i^r. It is clear that

$$\tau_i^r(x_j) = \tau_i^{o+}(x_j) = \sigma_{ik}^{o+}(x_j)n_k^r(x_j) , \qquad (x_j \in S^r), \tag{19}$$

where the superscripts + and - refer to the boundary values of the related quantities as the surface S^r is approached from outside (the positive side) and from inside (the negative side), respectively. The equilibrium considerations for the homogeneous region (D_0+D+S) subjected to the layer of body forces $T_i^{n^r}$ on S_r, $(r = 1,\ldots,m)$ and surface tractions $T_i^{n^o}$ on S_o require that

$$\tau_i^{o+} - \tau_i^{o-} + T_i^{n^r} = 0 , \quad (x_j \in S_r;\ i = 1,2,3;\ r = 1,\ldots,m). \tag{20}$$

Also, from the solution of the problem for the simply-connected domain D_r (see (17)) we have

$$\tau_i^{r-} = \sigma_{ij}^{r-}n_j^r = - T_i^{n^r} , \qquad (x_k \in S_r). \tag{21}$$

On the other hand, if $\nu_r = \nu_o$, $(r = 1,\ldots,m)$, using the equality of the displacements in D_r, $u_i^r(x_j) = u_i^o(x_j)$, $(x_j \in D_r,\ r = 1,\ldots,m)$, from the stress displacement relations it may easily be shown that

$$\frac{1}{\Delta\mu_r}\,\sigma_{ij}^r(x_\ell) = \frac{1}{\mu_o}\,\sigma_{ij}^o(x_\ell)\ , \qquad (x_\ell \in D_r,\ r = 1,\ldots,m), \tag{22}$$

or

$$\frac{1}{\Delta\mu_r}\,\tau_i^{r-}(x_j) = \frac{1}{\mu_o}\,\tau_i^{o-}(x_j)\ , \qquad (x_j \in D_r). \tag{23}$$

Thus, from (19), (20), (21), and (23) the components of the contact stress vector on S_r may be obtained as

$$\tau_i^r(x_j) = \tau_i^{o+}(x_j) = -\frac{\mu_r}{\mu_r - \mu_o}\,T_i^{n^r}(x_j)\ ,$$
$$(x_j \in S_r;\ i,j = 1,2,3;\ r = 1,\ldots,m). \tag{24}$$

Once the problem for the m+1 simply-connected domains is solved, noting that the displacement components u_i, $(i = 1,2,3)$ in the actual inclusion with elastic constants λ_r and μ_r are given by

$$u_i(x_j) = u_i^o(x_j) = u_i^r(x_j)\ ,$$
$$(x_j \in D_r;\ i,j = 1,2,3;\ r = 1,\ldots,m), \tag{25}$$

and (because of $\nu_r = \nu_o$, $(r = 1,\ldots,m)$)

$$\frac{\lambda_r}{\mu_r} = \frac{\lambda_o}{\mu_o} = \frac{\Delta\lambda_r}{\Delta\mu_r}\ , \tag{26}$$

the stresses in the actual inclusion may be expressed as

$$\sigma_{ij}(x_\ell) = \mu_r(u_{i,j} + u_{j,i}) + \lambda_r u_{k,k}\delta_{ij}$$
$$= \sigma_{ij}^o(x_\ell) + \sigma_{ij}^r(x_\ell)\ , \qquad (x_\ell \in D_r;\ r = 1,\ldots,m;\ i,j,\ell = 1,2,3), \tag{27}$$

where σ^{o}_{ij}, $(i,j=1,2,3)$ are the stress components in the matrix (D_0+D+S) which has the elastic constants μ_0, λ_0, and σ^{r}_{ij}, $(i,j=1,2,3)$ are the stress components in the auxiliary inclusion D_r with the elastic constants $\Delta\mu_r$, $\Delta\lambda_r$, $(r=1,\ldots,m)$.

Note 1. The results found in this section remain valid for the plane and axisymmetric problems, with the additional simplification that for $\nu_r = \nu_0$, $(r=1,\ldots,m)$ the resulting integral equations for the unknown functions T^{nr}_i, $(i=1,2;\ r=1,\ldots,m)$ would be one-dimensional.

Note 2. In the corresponding "antiplane shear" problem

$$u_2 = 0 = u_3, \quad u_{1,1} = 0, \quad \sigma_{1j} = \mu u_{1,j}, \quad T^{n}_{j} = 0, \quad (j=2,3), \tag{28}$$

and the results found in this section regarding the vanishing of the body forces F^{r}_{i} remain valid without any restriction on the elastic constants μ_r, $(r=0,1,\ldots,m)$. In this case (16 - 18) and (24) (with (25)) give the exact solution. In this problem too the interface S_r may be represented by a closed plane curve and the resulting integral equations (for T^{nr}_{1}, $r=1,\ldots,m$) are one-dimensional, the arc length measured along S_r being the variable.

3. THE FILAMENT MODEL

Let the filament be represented by a cylindrical inclusion of length 2c, radius r_0, and the elastic constants E_f, ν_f. Let the elastic constants of the surrounding matrix be E, ν. It is assumed that

(a) $\nu_f = \nu$;

(b) the dimensions of the matrix are large in comparison with c;

(c) the external load is the traction $\sigma_{zz} = \sigma_0$ applied to the matrix away from and parallel to the filament; and

(d) the length of the filament, 2c, is large in comparison with its diameter $2r_0$. Thus, the following basic relations for the infinite medium may be used in deriving the Green's functions

for the matrix [5]:

$$u_i = \frac{A}{\rho} X_i + \frac{B(x_i - \tau_i)}{\rho^3} \sum_1^3 (x_j - \tau_j) X_j \, , \qquad (i = 1,2,3) , \tag{29}$$

$$\rho^2 = \sum_1^3 (x_i - \tau_i)^2 \, ,$$

$$A = \frac{(1+\nu)(3-4\nu)}{8\pi E(1-\nu)} \, , \qquad B = \frac{1+\nu}{8\pi E(1-\nu)} \, , \tag{30}$$

where u_i, $(i = 1,2,3)$ are the components of the displacement vector at the point x_i due to the concentrated body forces, X_j acting at the point τ_j, $(j = 1,2,3)$, and x_i and τ_j refer to the rectangular coordinates. If we deal with an axisymmetric problem in which, referred to the cylindrical coordinates r,θ,z, the body forces R,Θ,Z are distributed over a ring $r = r_o$, $0 \le \theta < 2\pi$, $z = t$ in such a way that $\Theta = 0$ and R and Z are independent of θ, integrating over the ring, from (29) the displacement components at a point $(r = r_o,\ 0 \le \theta < 2\pi,\ z)$ may be obtained as

$$u_r(r_o,z) = K_{11}(z,t)R + K_{12}(z,t)Z \, ,$$

$$u_z(r_o,z) = K_{21}(z,t)R + K_{22}(z,t)Z \, ,$$

$$u_\theta = 0 \, , \tag{31.a-c}$$

$$K_{11}(z,t) = \frac{2A}{\rho_o}\left[2r_o + \frac{(1+\gamma)(t-z)^2}{r_o}\right] K(k)$$

$$- \frac{2A}{r_o}\left[\rho_o + \frac{\gamma}{\rho_o}\left(2r_o^2 + (t-z)^2\right)\right]E(k) \, ,$$

$$K_{12}(z,t) = -K_{21}(z,t) = -\frac{2A\gamma(t-z)}{\rho_o}\left[K(k) - E(k)\right] \, ,$$

$$K_{22}(z,t) = \frac{4Ar_o}{\rho_o}\left[K(k) + \gamma E(k)\right] \, , \tag{32.a-c}$$

$$\gamma = \frac{B}{A} = \frac{1}{3-4\nu} \;, \qquad \rho_o^2 = 4r_o^2 + (t-z)^2 \;, \qquad k = \frac{2r_o}{\rho_o} \;, \tag{33}$$

where $K(k)$ and $E(k)$ are complete elliptic integrals of the first and the second kind, respectively. A list of integrals used in the derivation of the kernels K_{ij}, $(i,j=1,2)$ may be found in Appendix A. Similar expressions may be obtained for

$u_r(r,\mp c)$, $u_z(r,\mp c)$ due to R,Z at (r_o,t),

$u_r(r,\mp c)$, $u_z(r,\mp c)$ due to R,Z at $(s,\mp c)$, and

$u_r(r_o,z)$, $u_z(r_o,z)$ due to R,Z at $(s,\mp c)$,

where $0 \leq (r,s) < r_o$, $-c < (z,t) < c$.

The filament model developed in this section will be used to study the stress state around the leading edge of a penny-shaped crack in the matrix located in the z=0 plane. Since $r_o \ll c$ and since the body forces R are locally self-equilibrating, the direct effect of R on the stress intensity factors along the crack periphery will be negligible. However, since the integral equations in R and Z will be coupled, the effect of R through Z may not be negligible. The first example discussed in this section will be devoted to study the effect of neglecting R on Z. For the sake of simplicity and in order to consider an extreme case, it will be assumed that the inclusion is rigid and the end effects are negligible. Thus, if the uniaxial tension $\sigma_{zz} = \sigma_o$ is the external load applied to the matrix away from the inclusion region (see the insert in Figure 2), from (31) the integral equations of the problem may be expressed by writing the displacement components along $(r = r_o$, $-c < z < c)$ equal to zero as follows:

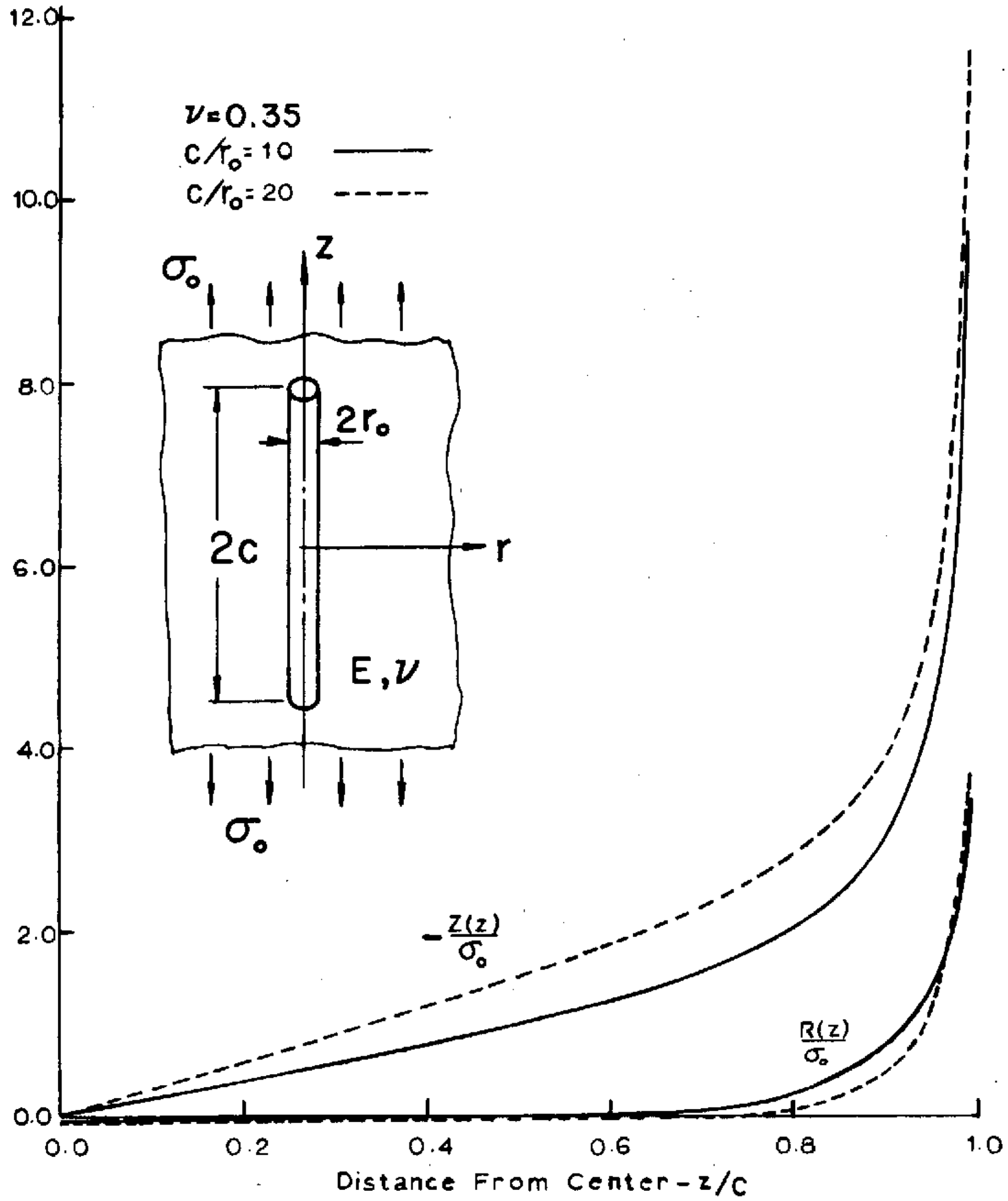


Figure 2. Radial and axial contact stresses for a rigid filament.

$$u_r(r_o,z) = -\frac{\nu r_o \sigma_o}{E} + \int_{-c}^{c} [K_{11}(z,t)R(t) + K_{12}(z,t)Z(t)]dt = 0 ,$$

$$u_z(r_o,z) = \frac{\sigma_o z}{E} + \int_{-c}^{c} [K_{21}(z,t)R(t) + K_{22}(z,t)Z(t)]dt = 0 ,$$

$$(-c < z < c), \qquad (34)$$

where K_{ij}, $(i,j = 1,2)$ are given by (32). A close examination of the kernels around $z = t$ would indicate that K_{11} and K_{22} have logarithmic singularities. This may be seen by observing that at $z = t$ $E(k)$ is finite and for small values of $|t-z|$ we have the following asymptotic relation:

$$K(k) = -\log|t-z| + \log 4\rho_o$$
$$+ \frac{1}{4}\frac{(t-z)^2}{\rho_o^2}[-\log|t-z| + \log 4\rho_o - 1] + \ldots \qquad (35)$$

Since the system (34) is of the first kind it is equivalent to a system of singular integral equations. In order to extract the correct behavior of the solution, it would be simpler to cast the system in the standard form with Cauchy-type singularities by formally differentiating the equations. Thus, separating the singular parts of the kernels, (34) becomes

$$\int_{-c}^{c} \frac{R(t)dt}{t-z} + \int_{-c}^{c} k_{11}(z,t)R(t)dt$$
$$- \frac{\gamma}{2r_o}\int_{-c}^{c}[\log|t-z| + k_{12}(z,t)]Z(t)dt = 0 ,$$

$$\frac{\gamma}{2r_o}\int_{-c}^{c}[\log|t-z| + k_{12}(z,t)]R(t)dt + \int_{-c}^{c}\frac{Z(t)dt}{t-z}$$
$$+ \int_{-c}^{c} k_{22}(z,t)Z(t)dt = -\frac{4\pi(1-\nu)\sigma_o}{(1+\nu)(3-4\nu)}, \quad (-c < z < c), \qquad (36)$$

where $\gamma = 1/(3-4\nu)$ and the bounded functions k_{ij}, $(i,j=1,2)$ are given by

$$k_{11}(z,t) = \frac{2r_0}{\rho_0}\left(\frac{\partial K(k)}{\partial z} - \frac{1}{t-z}\right) + \frac{2r_0-\rho_0}{\rho_0(t-z)} + \frac{(1+\gamma)(t-z)^2}{r_0\rho_0}\frac{\partial K(k)}{\partial z}$$

$$+ \frac{t-z}{\rho_0}K(k)\left[\frac{2r_0}{\rho_0^2} - \frac{2(1+\gamma)}{r_0} + \frac{(1+\gamma)(t-z)^2}{r_0\rho_0^2}\right]$$

$$- \frac{1}{r_0}\frac{\partial E(k)}{\partial z}\left[\rho_0 + \frac{\gamma}{\rho_0}\{2r_0^2 + (t-z)^2\}\right]$$

$$+ \frac{t-z}{r_0\rho_0}E(k)\left[1 + 2\gamma - \frac{\gamma}{\rho_0^2}\{2r_0^2 + (t-z)^2\}\right] ,$$

$$k_{12}(z,t) = k_{21}(z,t) = \frac{\gamma}{2r_0\rho_0}(2r_0 - \rho_0)\log|t-z|$$

$$- \frac{\gamma}{\rho_0}[K(k) + \log|t-z|] + \frac{\gamma}{\rho_0}E(k)$$

$$+ \frac{\gamma(t-z)}{\rho_0^3}[K(k) - E(k)] + \frac{\gamma(t-z)}{\rho_0}\left[\frac{\partial K(k)}{\partial z} - \frac{\partial E(k)}{\partial z}\right],$$

$$k_{22}(z,t) = \frac{2r_0}{\rho_0}\left[\frac{\partial K(k)}{\partial z} - \frac{1}{t-z}\right] + \frac{2r_0-\rho_0}{\rho_0(t-z)} + \frac{2r_0(t-z)}{\rho_0^3}K(k)$$

$$+ \frac{2r_0\gamma}{\rho_0}\frac{\partial E(k)}{\partial z} + \frac{2r_0\gamma(t-z)}{\rho_0^3}E(k) , \qquad (37.a-c)$$

$$\frac{\partial K(k)}{\partial z} = \frac{E(k)}{t-z} - \frac{t-z}{\rho_0^2}K(k) ,$$

$$\frac{\partial E(k)}{\partial z} = \frac{t-z}{\rho_0^2}[E(k) - K(k)] , \qquad k^2 = \frac{4r_0^2}{4r_0^2 + (t-z)^2} . \qquad (38)$$

Referring to [6] it may be shown that the solutions of (36), R and Z, have integrable singularities at $\mp c$, the index of the system is $\kappa = 1$, and hence the general solution will contain two

arbitrary constants. On the other hand (36) states that the z-derivatives of the displacements u_r and u_z rather than u_r and u_z are zero along $(r = r_0, -c < z < c)$. Thus, the solution of (36) must satisfy a set of single-valuedness conditions which may then be used to determine the arbitrary constants resulting from the general solution. These conditions may be expressed by fixing u_r and u_z at any point along the line of integration, say, for example at $z = 0$, giving

$$u_r(r_0,0) = 0 \ , \qquad u_z(r_0,0) = 0 \ , \tag{39.a,b}$$

where the expressions for u_r and u_z are given by (34).

Considering the symmetry of the problem, the solution of the system of singular integral equations (36) subject to the conditions (39) is of the following form [6]:

$$R(z) = F(z)(c^2 - z^2)^{-1/2} \ , \quad Z(z) = G(z)(c^2 - z^2)^{-1/2} , \tag{40.a,b}$$

where $F(z) = F(-z)$ and $G(z) = -G(-z)$ are bounded functions which may easily be obtained numerically (e.g., [7,8]). Some numerical results obtained from (36) are shown in Figures 2 - 4. Figure 2 gives $R(z)$ and $Z(z)$ for $\nu = 0.35$, $(c/r_0) = 10$ and $(c/r_0) = 20$. Figure 3 shows the effect of the Poisson's ratio ($\nu = 0.2$ and $\nu = 0.35$ used in the Figure roughly correspond to a glass and an epoxy matrix, respectively). From the viewpoint of this study aiming to simplify the filament model the important result is shown in Figure 4. Here the body force $Z(z)$ (which, in this case is also the contact stress) is given as obtained from (36) with and without neglecting $R(z)$. It appears that for the practical range of c/r_0 ratios the effect of neglecting R on Z will be

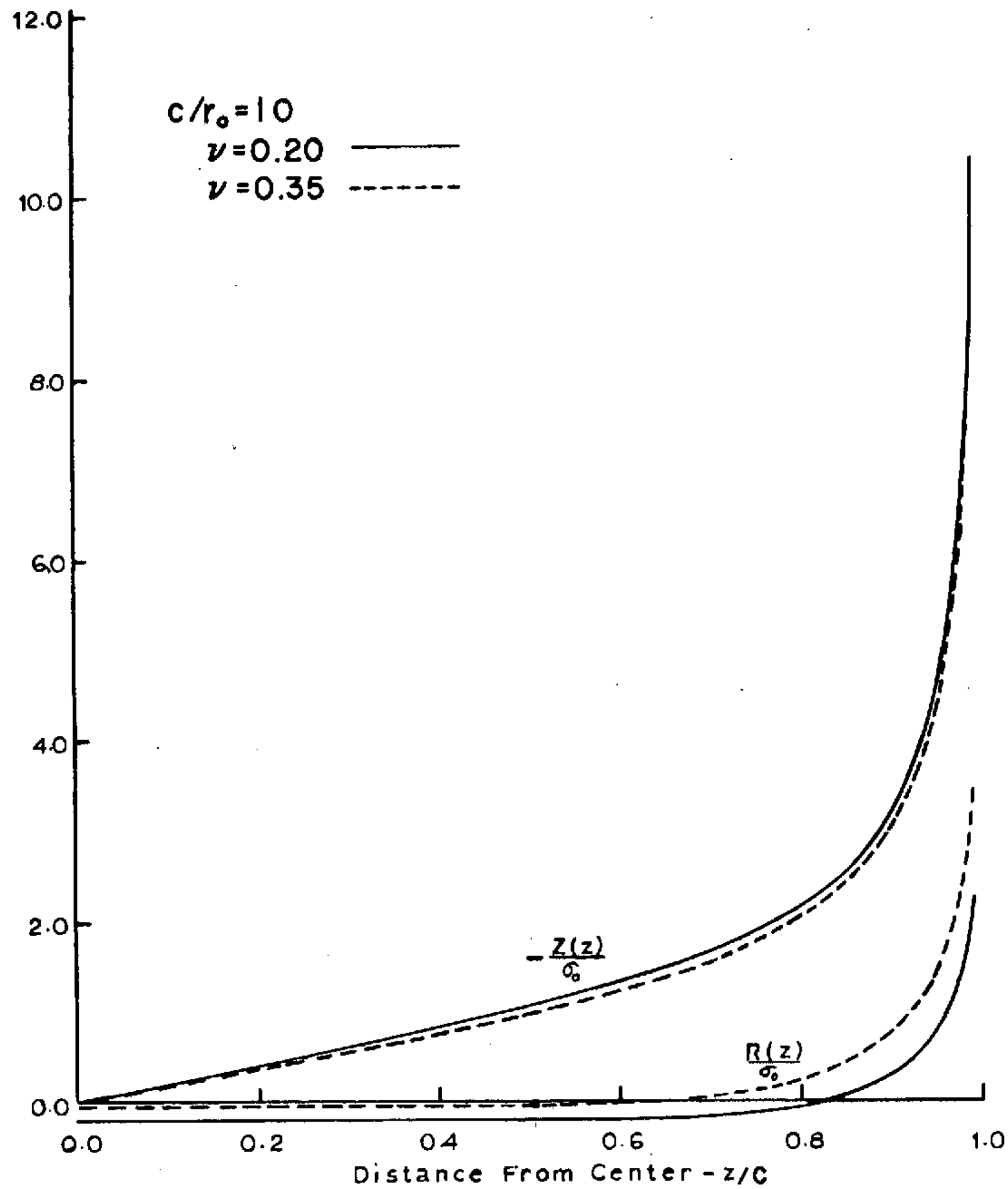


Figure 3. The effect of Poisson's ratio on the contact stresses for a rigid filament.

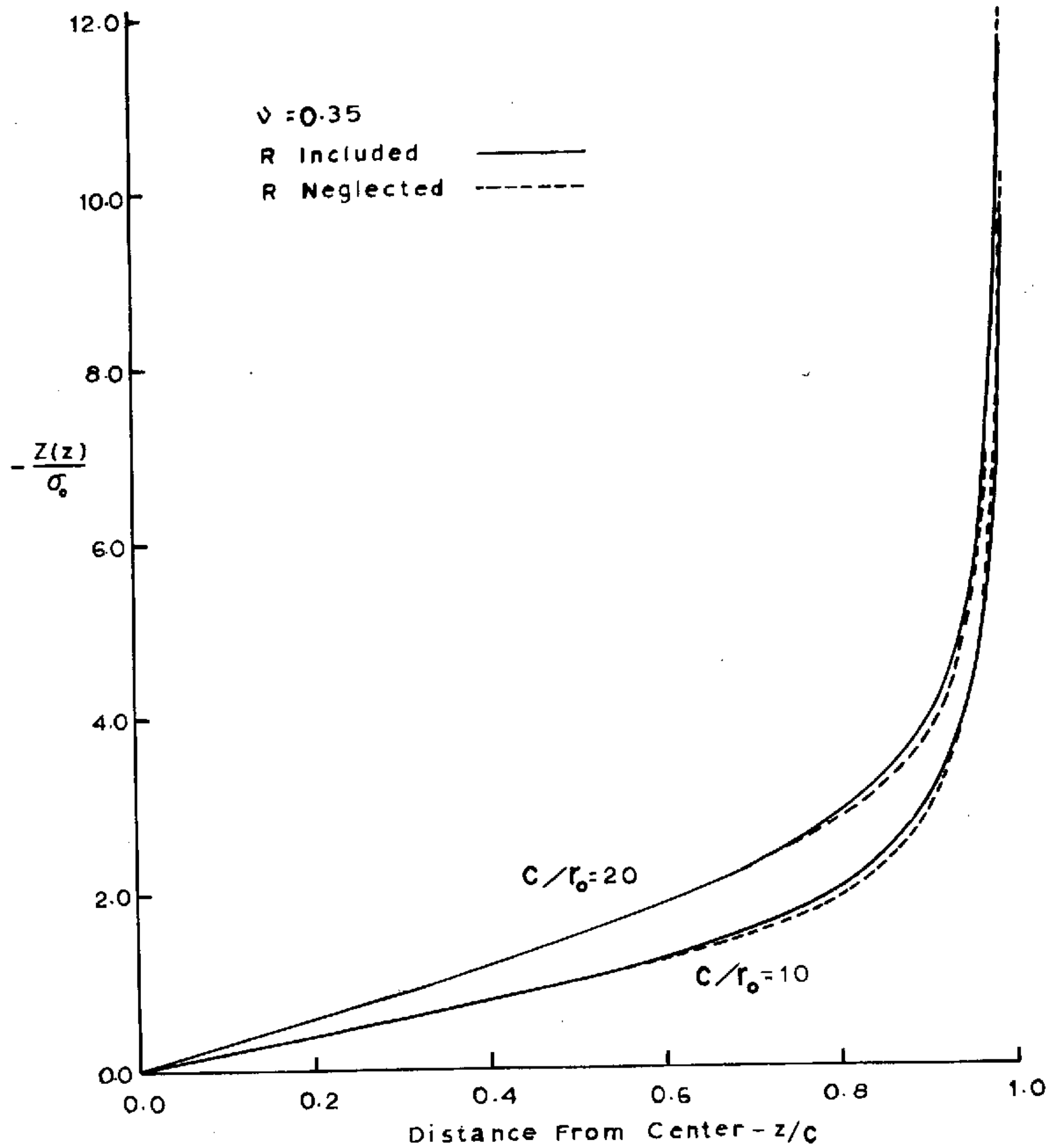


Figure 4. The effect of the radial body force on the axial contact stress in a rigid filament.

negligible. Hence, for the remainder of this study the radial component $R(z)$ of the body force will be neglected.

In the case of the elastic filament, in order to simplify the resulting system of integral equations, in addition to neglecting the radial component R of the body force, it will be assumed that the body force $Z(r,\mp c)$, $(0 \leq r < r_0)$ at the ends is uniformly distributed. Again, the effect of this assumption will be local and will be negligible on the stresses in the matrix in $z=0$ plane (and hence, on the stress intensity factor along the leading edge of the crack when the crack problem is considered). Thus, the unknown quantities will be the distributed body force $Z(r_0,z)$, $(-c < z < c)$ and the constant $Z(r,c) = p = -Z(r,-c)$. These quantities will be determined from the integral equation and the algebraic equation obtained by matching the displacements of the matrix and the auxiliary filament (with elastic constants $E_a = E_f - E$ and ν) along the surface $(r = r_0,\ -c < z < c)$ and at an appropriate point at the end $z=c$ (which will be selected as $r=0,\ z=c$).

Due to the large length-to-diameter ratio c/r_0, the filament will be approximated by a one-dimensional body subjected to body forces $-\frac{2}{r_0} Z(z)$ distributed uniformly over the cross-section $(0 \leq r < r_0,\ z)$ and the end tractions $-p$ distributed again uniformly over the ends $z = \mp c$. Thus, the displacement in the filament may be expressed as

$$u_{fz}(z) = -u_{fz}(-z) = -\frac{pz}{E_f - E} - \frac{2}{r_0(E_f - E)} \int_0^z dt \int_t^c Z(\eta)d\eta\ ,$$

$$(0 \leq z < c), \qquad (41)$$

$$u_{fz}(c) = -\frac{1}{E_f - E}\left[pc + \frac{2}{r_o}\int_0^c tZ(t)dt\right] . \tag{42}$$

Evaluating now the displacements $u_z(r_0,z)$, $(-c < z < c)$ and $u_z(0,c)$ in the matrix due to the body forces $Z(r_0,t)$, $(-c < t < c)$ and $Z(r,c) = -Z(r,-c) = p$, $(0 \leq r < r_0)$, and the traction at infinity $\sigma_{zz} = \sigma_0$ we obtain

$$u_z(r_0,z) = \frac{\sigma_0 z}{E} + \int_{-c}^{c} K_{22}(z,t)Z(t)dt - Ap[M_1(z) - N_1(z)$$

$$+ \gamma\{(c+z)^2 M_2(z) - (c-z)^2 N_2(z)\}] ,$$

$$u_z(0,c) = \frac{\sigma_0 c}{E} + 2\pi r_0 A \int_{-c}^{c} \left(c + \frac{\gamma(c-t)^2}{r_0^2 + (c-t)^2}\right) \frac{Z(t)dt}{[r_0^2 + (c-t)^2]^{1/2}}$$

$$+ 2\pi A C_2 pc , \tag{43.a,b}$$

where

$$\gamma = \frac{1}{3 - 4\nu} , \qquad A = \frac{(1+\nu)(3 - 4\nu)}{8\pi E(1-\nu)} ,$$

$$C_2 = 2 + \frac{r_o}{c} - (4 + r_0^2/c^2)^{1/2} + 4\gamma\left(\frac{1}{(4 + r_0^2/c^2)^{1/2}} - \frac{1}{2}\right) , \tag{44}$$

$$M_i(z) = \int_0^{r_o} rdr \int_0^{2\pi} \frac{d\theta}{[r^2 + r_0^2 - 2rr_0\cos\theta + (c+z)^2]^{(2i-1)/2}} ,$$

$$N_i(z) = \int_0^{r_o} rdr \int_0^{2\pi} \frac{d\theta}{[r^2 + r_0^2 - 2rr_0\cos\theta + (c-z)^2]^{(2i-1)/2}} ,$$

$$(i = 1,2,\ldots), \tag{45.a,b}$$

and K_{22} is given by (32.c). Again, differentiating (41) and (43.a) and using (42) and (43.b), from the conditions of continuity of the displacements we find

$$\int_{-c}^{c} \frac{Z(t)dt}{t-z} + \int_{-c}^{c} k_{22}(z,t)Z(t)dt + \frac{2C_1 E}{r_o(E_f - E)} \int_{z}^{c} Z(t)dt$$

$$+ \frac{p}{2} [k_2(z) + \frac{2C_1 E}{E_f - E}] = - C_1 \sigma_o , \qquad (-c < z < c),$$

$$\int_{-c}^{c} k_1(t)Z(t)dt + \frac{2C_1 E}{r_o(E_f - E)} \int_{0}^{c} tZ(t)dt$$

$$+ pc(C_2 + \frac{C_1 E}{E_f - E}) = - C_1 c \sigma_o , \qquad (46.a,b)$$

where

$$C_1 = \frac{4\pi(1-\nu)}{(1+\nu)(3-4\nu)} ,$$

the kernel $k_{22}(z,t)$ is given by (37.c) and

$$k_1(t) = \frac{\pi r_o}{[r_o^2 + (c-t)^2]^{1/2}} (1 + \frac{(c-t)^2}{(3-4\nu)[r_o^2 + (c-t)^2]}),$$

$$k_2(z) = (1 - \frac{2}{3-4\nu})[(c+z)M_2(z) + (c-z)N_2(z)]$$

$$+ 3[(c+z)^3 M_3(z) + (c-z)^3 N_3(z)] . \qquad (47.a,b)$$

The functions M_i and N_i appearing in (47.b) are defined by (45). Some simplifications for the evaluation of these functions may be found in Appendix B.

The integral equation (46.a) and the algebraic equation (46.b) determine the function Z(z) and the constant p. Noting that at $z = 0$ K_{22} is an even function of t, from (43.a), (45) and (41) it is seen that the single-valuedness condition $u_z(r_o,0) - u_{fz}(0) = 0$ will be automatically satisfied provided the solution of (46) is restricted to a class of odd functions (as required by the symmetry of the problem), i.e., $Z(t) = -Z(-t)$, $(-c < t < c)$, and $Z(r,c) = p = -Z(r,-c)$, $(0 \le r < r_o)$. The numerical

solution of (46) may again be obtained in a straightforward way [7,8].

Once $Z(z)$ and p are obtained all the desired field quantities may be evaluated in terms of definite integrals having the related Green's functions as kernels and Z and p as density functions. In fracture studies, of particular interest are the contact shear $\sigma^{+}_{rz}(r_0,z)$ along the filament-matrix interface and the axial stress $\sigma_{fzz}(z)$ in the filament. The general expression for the contact stress is given by (24), which in this case becomes

$$\sigma^{+}_{rz}(r_0,z) = -\frac{E_f}{E_f - E}\, Z(z) \ . \tag{48}$$

The general expression for the stresses in the filament is given by (27), namely

$$\sigma_{fzz}(r,z) = \sigma_{zz}(r,z) + \sigma_{azz}(z) \ , \quad (0 \le r < r_0,\ |z| < c), \tag{49}$$

where σ_{zz} is the stress in the matrix due to the external loads σ_0, $Z(z)$, and p, and σ_{azz} is the axial stress in the auxiliary filament which has the elastic constants $E_a = E_f - E$ and ν. σ_{zz} appearing in (49) may be obtained by adding σ_0 to the stress component σ_{zz} evaluated from (29) and the related stress-displacement relations. Here, since r_0 is relatively very small and since the auxiliary filament is approximated by a one-dimensional bar, the r-dependence of σ_{zz} will be neglected and it will be represented by its value at $r = 0$. The stress in the auxiliary filament may easily be obtained from (41) as

$$\sigma_{azz}(z) = -p - \frac{2}{r_0}\int_z^c Z(t)dt \ . \tag{50}$$

Thus, after some manipulations the axial stress in the filament is found to be

$$\sigma_{fzz}(z) = \sigma_0 - ph_1(z) - \frac{2}{r_0}\int_z^c Z(t)dt + \int_{-c}^{c} h_2(z,t)Z(t)dt \ ,$$

$$(0 \leq z < c), \qquad (51)$$

where

$$h_1(z) = \frac{1}{4(1-\nu)}\left[(1-2\nu)\left(\frac{c-z}{[r_0^2+(c-z)^2]^{1/2}} + \frac{c+z}{[r_0^2+(c+z)^2]^{1/2}}\right)\right.$$

$$\left. + \frac{(c-z)^3}{[r_0^2+(c-z)^2]^{3/2}} + \frac{(c+z)^3}{[r_0^2+(c+z)^2]^{3/2}}\right] ,$$

$$h_2(z,t) = \frac{r_0(t-z)}{4(1-\nu)[r_0^2+(t-z)^2]^{3/2}}\left[1 - 2\nu + \frac{3(t-z)^2}{r_0^2+(t-z)^2}\right]. \quad (52.a,b)$$

The results of a numerical example giving the filament stress are shown in Figures 5 and 6. Figure 5 shows $\sigma_{fzz}(z)$ for various combinations of c/r_0 and E_f/E.

For large c/r_0 ratios it is reasonable to expect that the relative contribution of the end tractions p (particularly away from the ends) would be negligible. Figure 6 shows the results obtained with and without ignoring the effect of p for various combinations of c/r_0 and E_f/E. It is clear from the figure that, in future calculations regarding the application of the filament model developed in this paper, the effect of the end tractions may indeed be ignored.

4. COMPARISON WITH OTHER MODELS

Two other possible models for an elastic filament are the

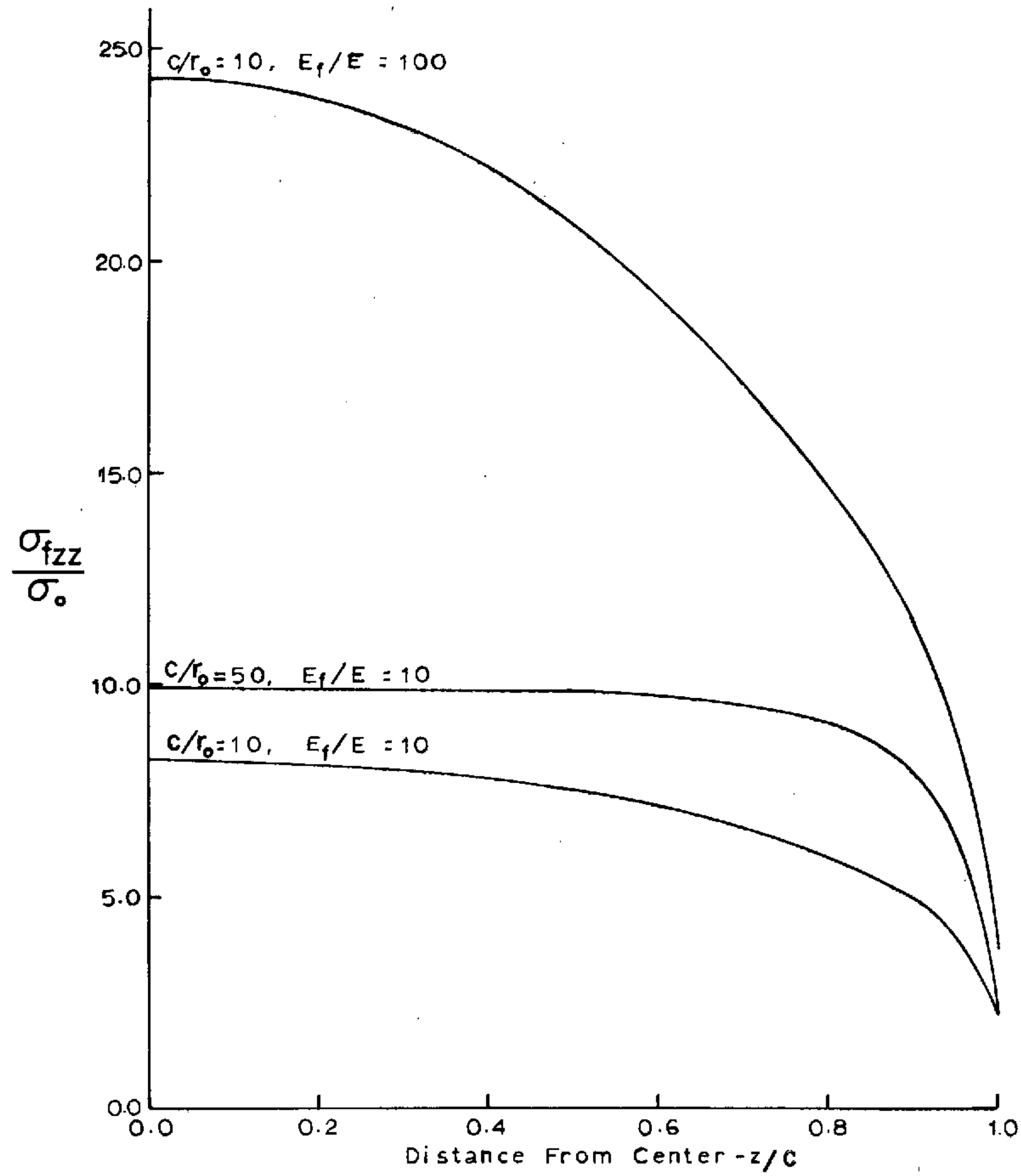


Figure 5. Axial stress in an elastic filament.

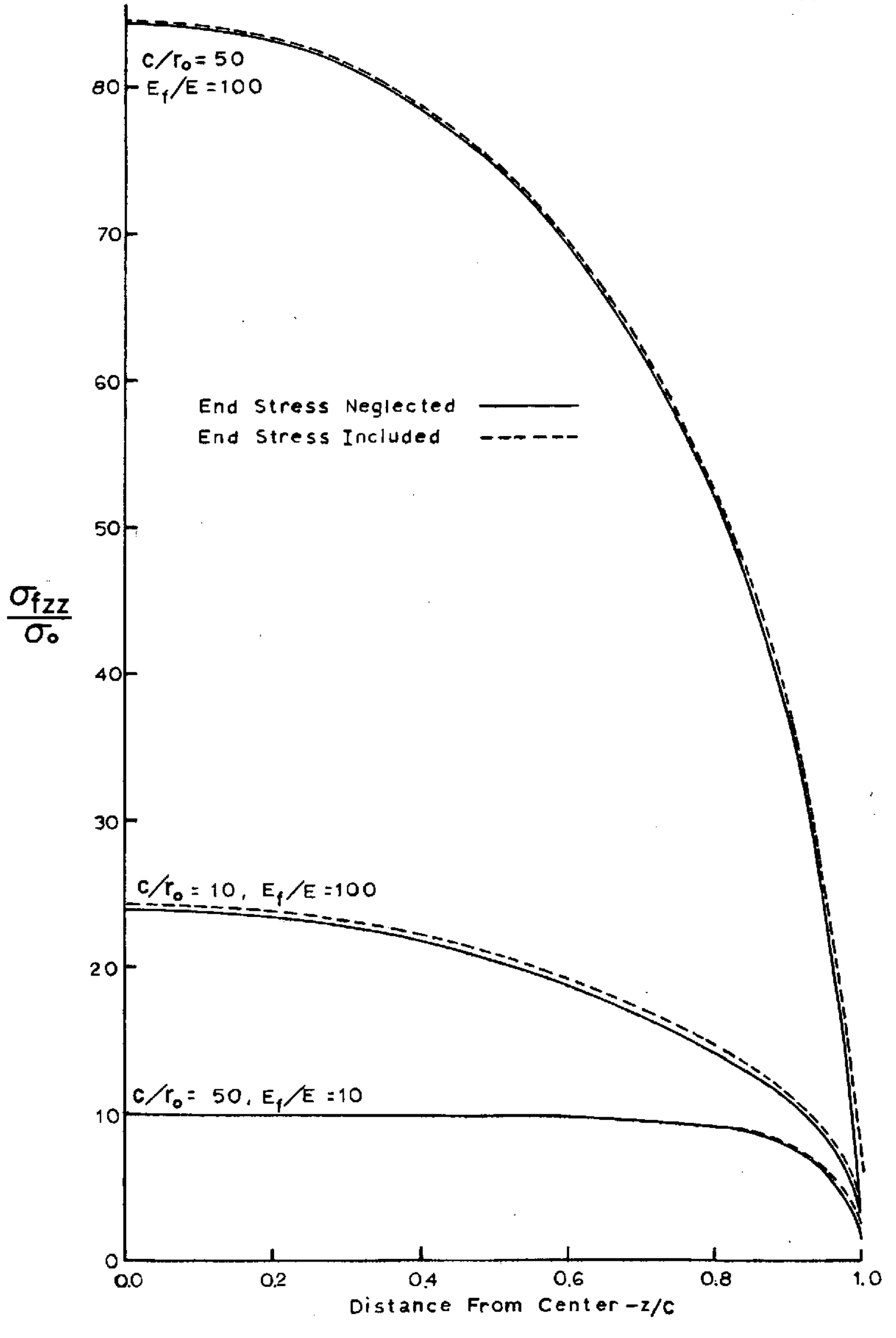


Figure 6. The effect of end tractions p on the axial stress in an elastic filament.

ellipsoidal inclusion considered in [1] and the model discussed in [2]. The solution given in [1] is in closed form where it is shown that the stress state in the inclusion is uniform. The expression for the stresses are rather lengthy and will not be presented in this paper*. The calculated results for the stresses $\sigma_{rr} = \sigma_{\theta\theta}$ and σ_{zz} in the inclusion (filament) which is in the form of an ellipsoid with the semi-axes (c, r_0, r_0) are shown in Figures 7 and 8. Figure 9 shows the comparison of the maximum filament stresses $\sigma_{fzz}(0)$ obtained from the ellipsoidal inclusion solution and from the elastic filament model given in the previous section (equation (51)). The agreement appears to be quite good for lower values of E_f/E and acceptable for higher E_f/E.

Following the procedure outlined in [2] the filament problem considered in this paper may be reduced to a Fredholm-type integral equation of the second kind with a logarithmic singularity. In this model the body force Z(z) acting on the matrix is assumed to be distributed uniformly over the cross-section (z = constant, $0 \leq r < r_0$) and the integral equation is obtained by matching the strains ε_{zz} in the matrix and in the auxiliary filament** (which is also assumed to be a one-dimensional bar). The calculated filament stresses obtained from this model for various combinations of c/r_0 and E_f/E are shown in Figure 10. Figure 11 shows the comparison of the filament stresses obtained from the models given in [1] and [2], and from that described in this paper.

---

*The details may be found in [9].

**The details of the derivation of the integral equation and the solution may be found in [9].

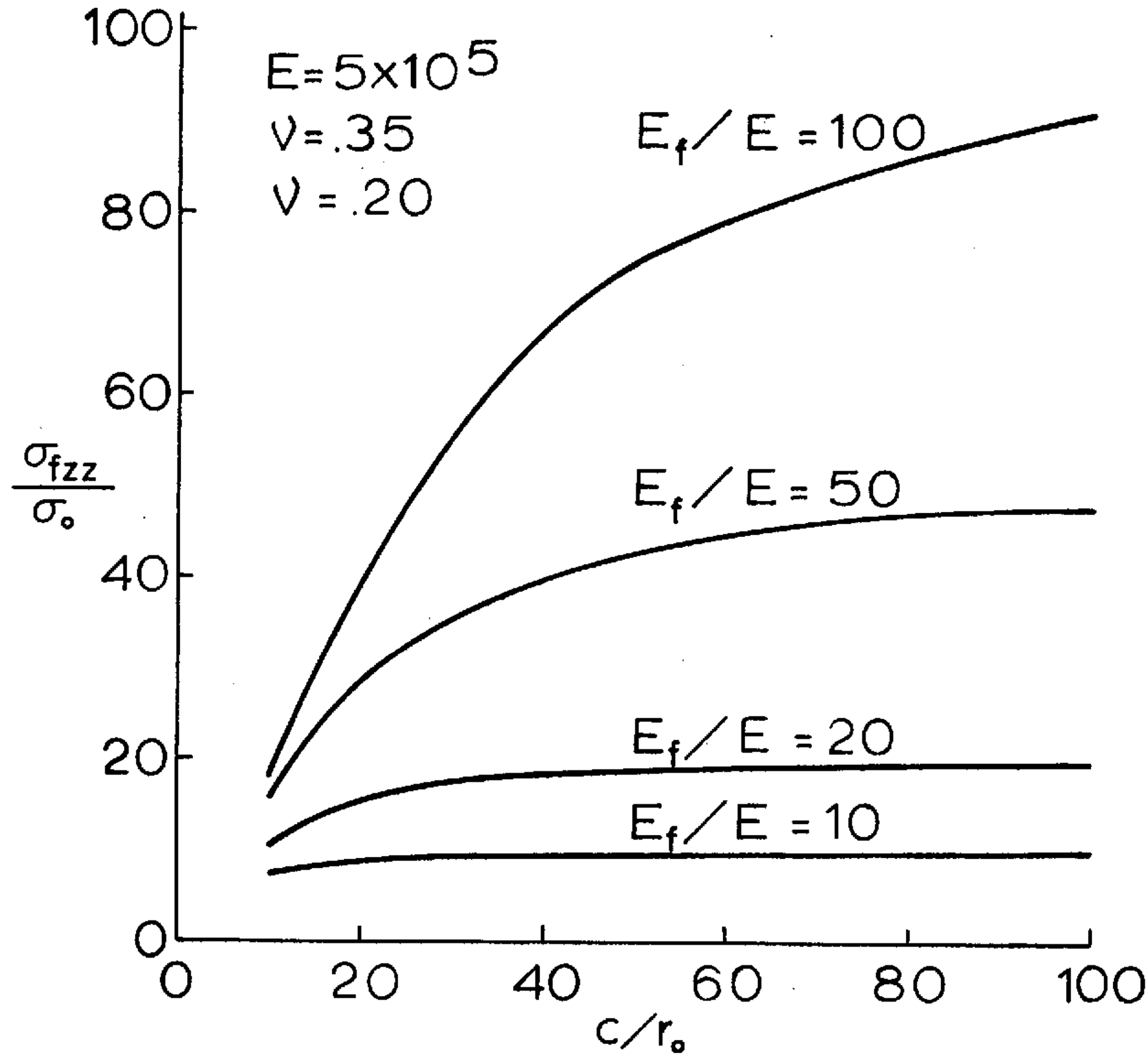


Figure 7. The axial stress in an elastic ellipsoidal inclusion [Ref. 1].

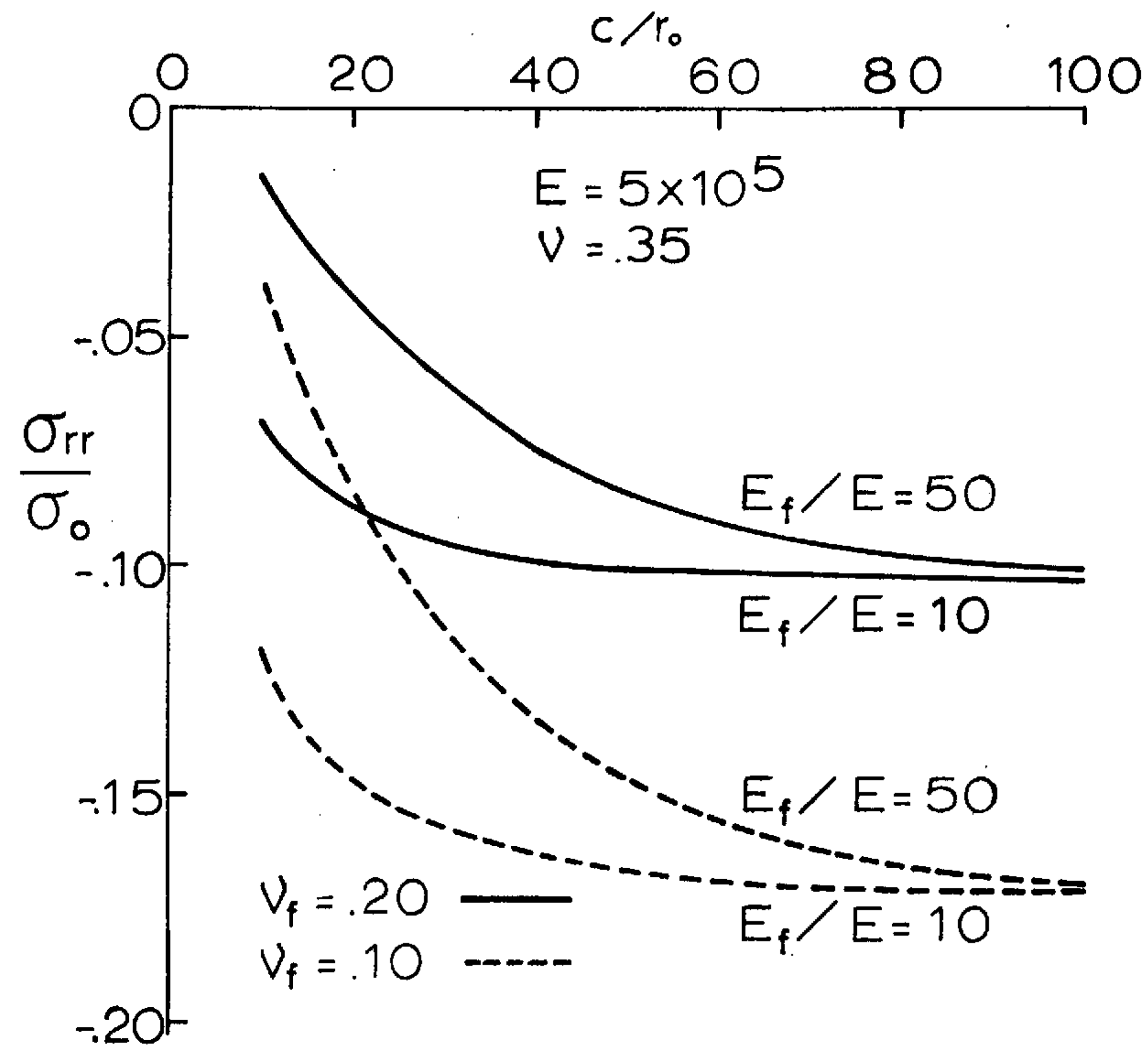


Figure 8. The radial stress in an elastic ellipsoidal inclusion.

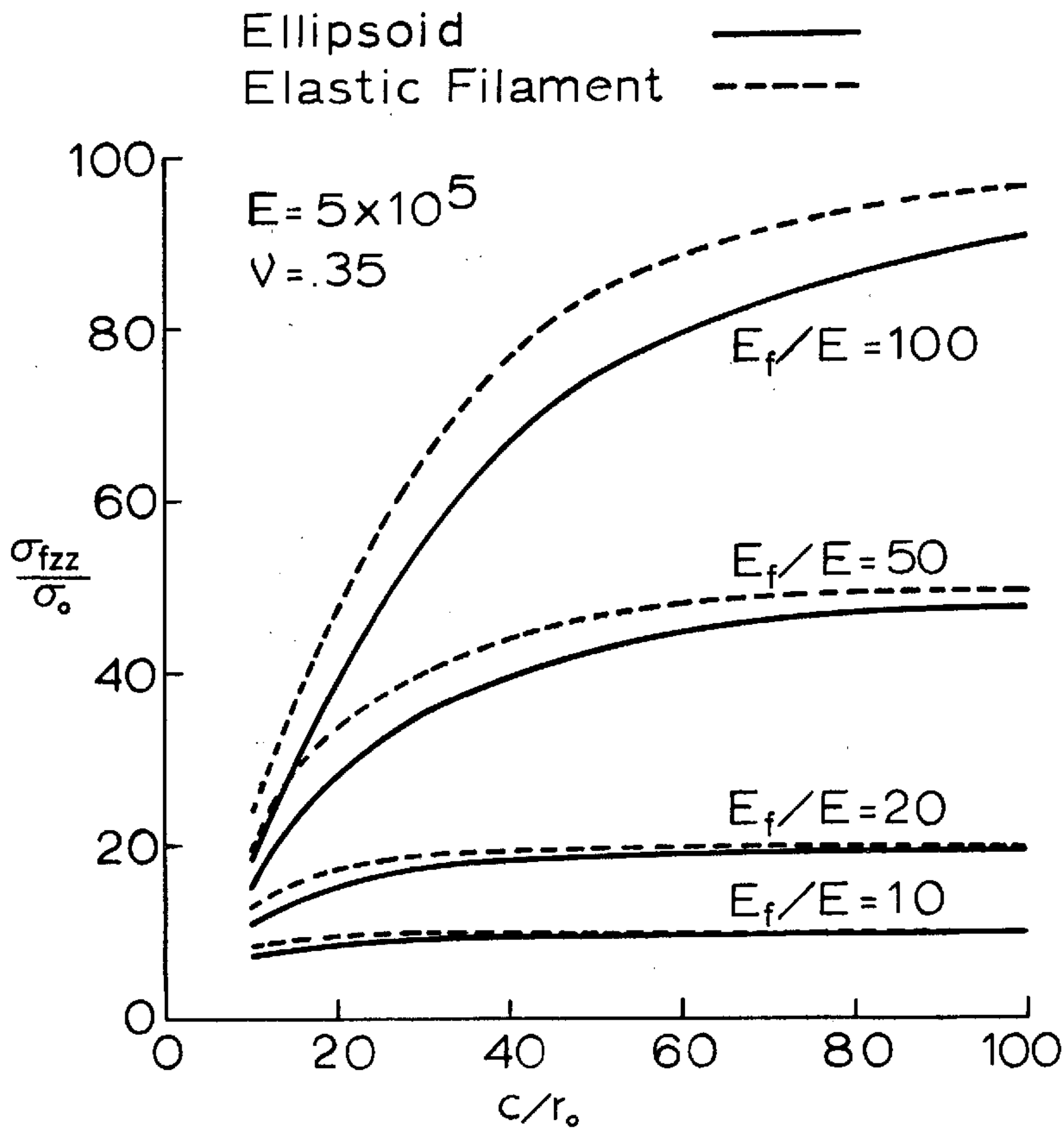


Figure 9. Comparison of the (maximum) axial stresses in an elastic ellipsoidal inclusion and in an elastic filament.

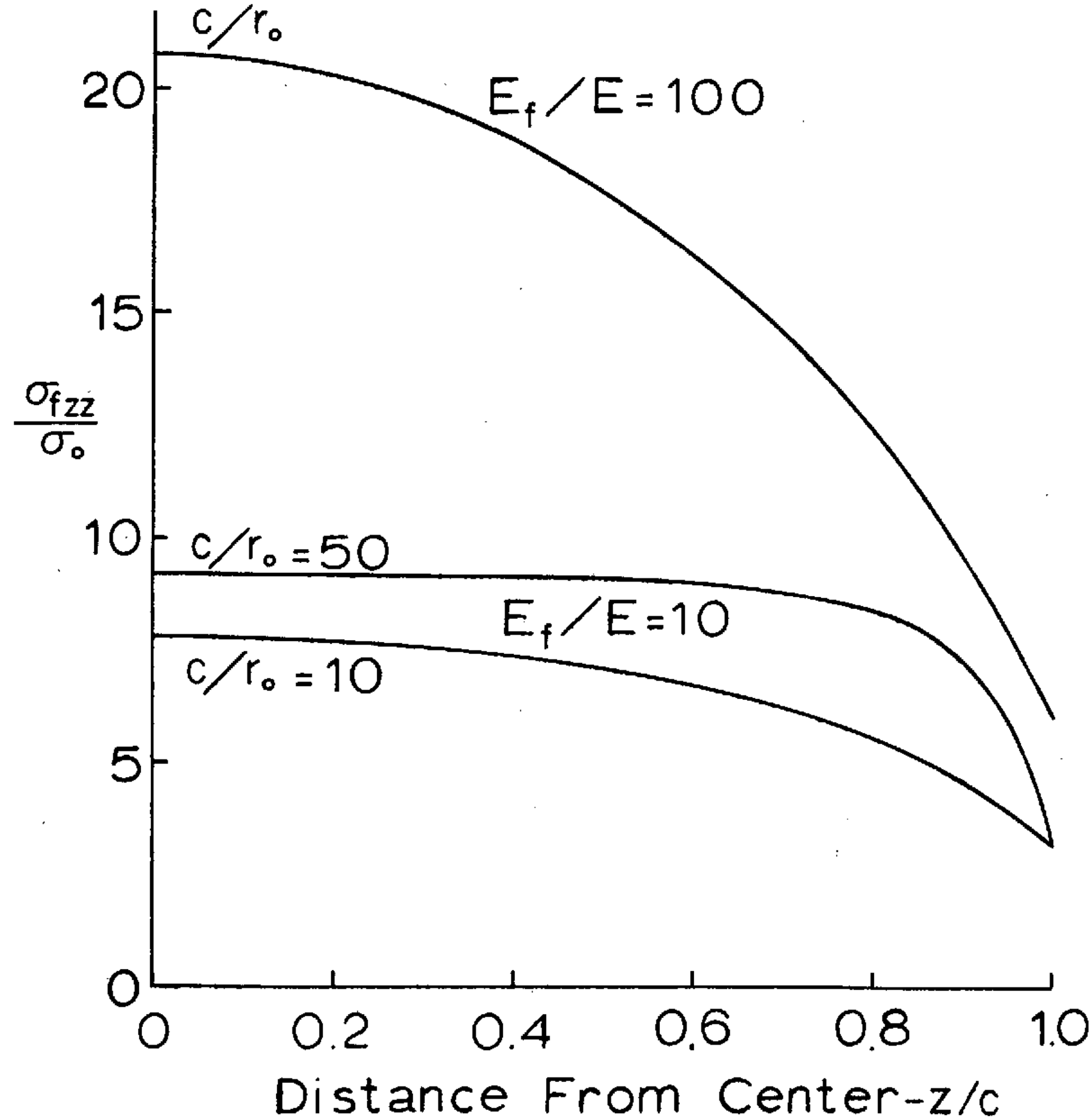


Figure 10. Axial stress in an elastic inclusion calculated from the model of Ref. 2.

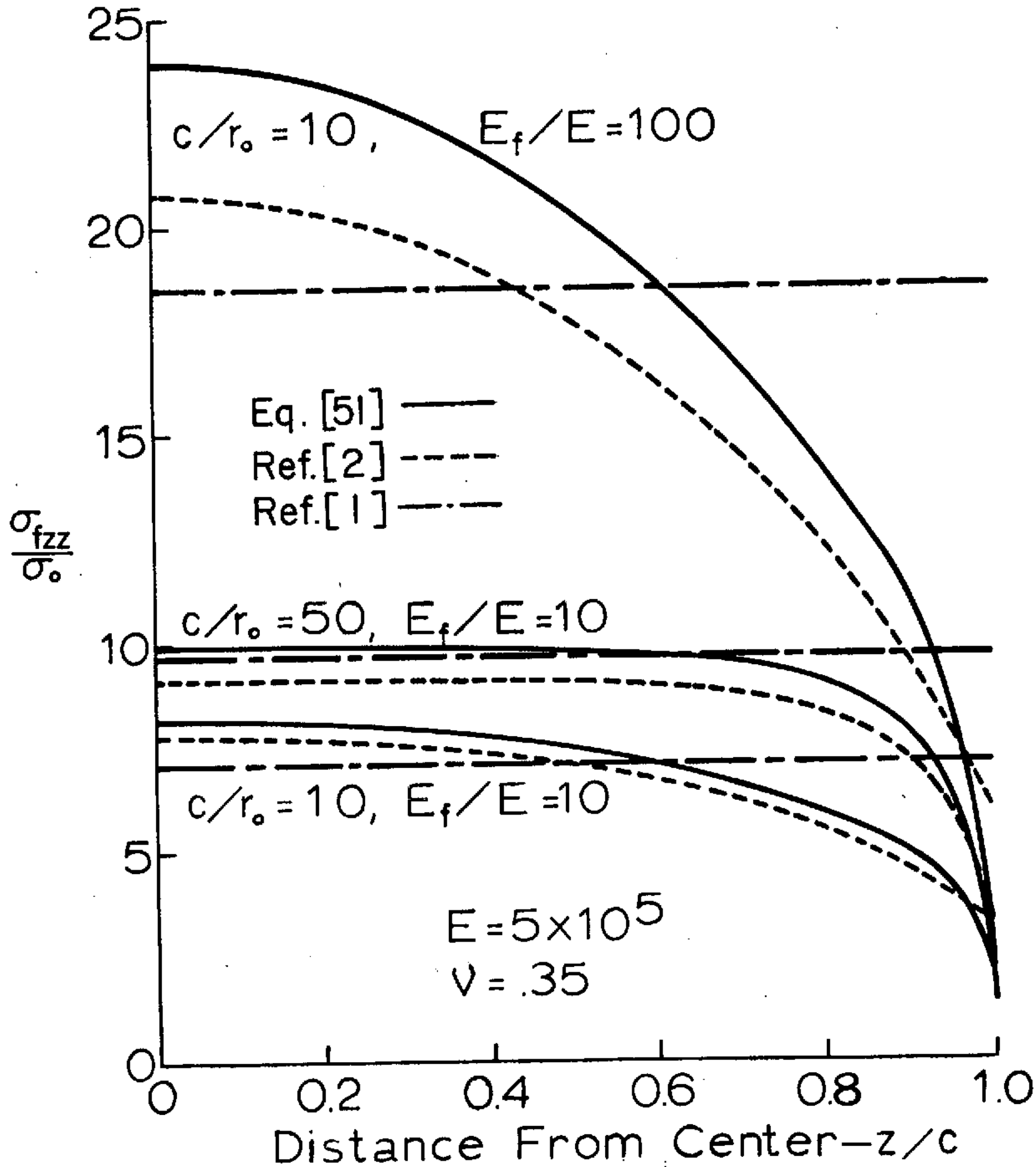


Figure 11. Comparison of the axial stresses obtained from three different filament models.

5. THE CASE OF MULTIPLE FILAMENTS

The application of the filament model developed in Section 3 of this paper to problems involving multiple filaments is straightforward provided the distance between any two filaments is sufficiently large compared to r_i, $(i = 1,\dots,N)$ (r_i being the radius of the ith filament) so that the variation of the body force $Z_i(r_i,z)$, $(i = 1,\dots,N)$ along the circumference of the filament may be neglected. In this case the problem may easily be shown to reduce to a system of N singular integral equations in the unknown functions $Z_i(z)$, $(i = 1,\dots,N)$ which may be solved numerically in a routine way [7,8]. In particular, if the filaments are identical and are located symmetrically the problem may be simplified considerably. This is the case where $r_1 = \dots = r_N = r_o$, $c_1 = \dots = c_N = c$, $z=0$ is a plane of symmetry for all filaments (which are parallel to the z axis), the filaments are evenly spaced on a circle of radius b on z=0 plane, and the matrix is again subjected to a uniaxial stress σ_o parallel to the z axis and away from the filament region (see the insert in Figure 12). For this special case because of symmetry $Z_1(z) = \dots = Z_N(z) = Z(z)$ and $p_1 = \dots = p_N = p$, and hence the problem reduces to a single integral equation and a single algebraic equation in Z and p. Referring to Section 3 of this paper, after some simple manipulations we find

$$\int_{-c}^{c} \frac{Z(t)dt}{t-z} + \int_{-c}^{c} m_{22}(z,t)Z(t)dt + \frac{2C_1E}{r_o(E_f - E)} \int_{z}^{c} Z(t)dt$$

$$+ \frac{p}{2}\left[m_2(z) + \frac{2C_1E}{E_f - E}\right] = - C_1\sigma_o \,, \qquad (-c < z < c),$$

$$\int_{-c}^{c} m_1(t)Z(t)dt + \frac{2C_1 E}{r_0(E_f - E)} \int_{0}^{c} tZ(t)dt$$

$$+ p(C_3 + \frac{C_1 E}{E_f - E}) = - C_1 c\sigma_0 , \qquad (53.a,b)$$

$$m_{22}(z,t) = k_{22}(z,t) + \pi r_0 (t-z) \sum_{i=2}^{N} \frac{1}{[d_i^2 + (t-z)^2]^{3/2}}$$

$$\times [1 - 2\gamma + \frac{3\gamma(t-z)^2}{d_i^2 + (t-z)^2}] ,$$

$$m_2(z) = k_2(z) + \pi r_0^2 \sum_{i=2}^{N} [\frac{c-z}{[d_i^2 + (c-z)^2]^{3/2}} (1 - 2\gamma + \frac{3\gamma(c-z)^2}{d_i^2 + (c-z)^2})$$

$$+ \frac{c+z}{[d_i^2 + (c-z)^2]^{3/2}} (1 - \gamma + \frac{3\gamma(c+z)^2}{d_i^2 + (c+z)^2})] ,$$

$$m_1(z) = k_1(z) + \pi r_0 \sum_{i=2}^{N} [\frac{1}{[d_i^2 + (c-z)^2]^{1/2}} (1 + \frac{\gamma(c-z)^2}{d_i^2 + (c-z)^2})] ,$$

$$C_3 = C_2 - \frac{\pi r_0^2}{2c^2} \sum_{i=2}^{N} [\frac{1}{(4 + d_i^2/c^2)^{1/2}} (1 + \frac{4\gamma}{4 + d_i^2/c^2}) - \frac{c}{d_i}] ,$$

$$C_1 = \frac{4\pi(1-\nu)}{(1+\nu)(3 - 4\nu)} ,$$

$$d_i = b\{2[1 - \cos(2\pi\{i-1\}/N)]\}^{1/2} , \quad (i = 2,\dots,N), \quad (54.a\text{-}f)$$

where r_0, c are the dimensions and E_f is the Young's modulus of the filaments, E and ν are the elastic constants of the matrix, $k_1(z)$ and $k_2(z)$ are given by (47), C_2 and γ are given by (44), $k_{22}(z,t)$ is given by (37.c), $Z(z)$ is the unknown body force along the filament-matrix interface ($r = r_0$, $|z| < c$), and p is the uniformly distributed body force applied to the ends ($0 \leq r < r_0$, $z = \mp c$).

Some of the numerical results obtained from the solution of (53) are given by Figures 12 - 14. Figure 12 shows the distribution of the axial stress in two symmetrically located filaments, where again the problem is solved with and without taking the end effect p into account. In this example, the effect of p is seen to be quite insignificant. Again, for N=2, the effect of length-to-diameter ratio and spacing of the filaments are shown in Figure 13. Note that as the distance b between the filaments decreases the axial stress in the filaments also decreases. Figure 14 shows the ratio of the maximum filament stress (which is at z=0) for N>1 to that for N=1 as a function of the distance parameter b (see insert in Figure 12) (Figure 14.a), and as a function of modulus ratio E_f/E (Figure 14.b). As expected, the interaction effect increases with increasing N and increasing E_f/E.

REFERENCES

1. Eshelby, J.D., "The determination of the elastic field of an ellipsoidal inclusion and related problems", Proc. Roy. Soc., Vol. 241, Ser. A, pp. 376-396 (1967).

2. Muki, R. and Sternberg, E., "Elastostatic load transfer to a half-space from a partially embedded axially loaded rod", Int. J. Solids, Structures, Vol. 6, pp. 69-90 (1970).

3. Goodier, J.N. and Hsu, C.S., "Transmission of tension from a bar to a plate", J. Appl. Mech., Vol. 21, Trans. ASME, pp. 147-150 (1954).

4. Muki, R. and Sternberg, E., "On the stress analysis of overlapping bonded elastic sheets", Int. J. Solids, Structures, Vol. 4, pp. 75-94 (1968).

5. Love, A.E.H., A Treatise on the Mathematical Theory of Elasticity, Dover Publ., New York (1944).

6. Muskhelishvili, N.I., Singular Integral Equations, Noordhoff N.V. Groningen, Holland (1953).

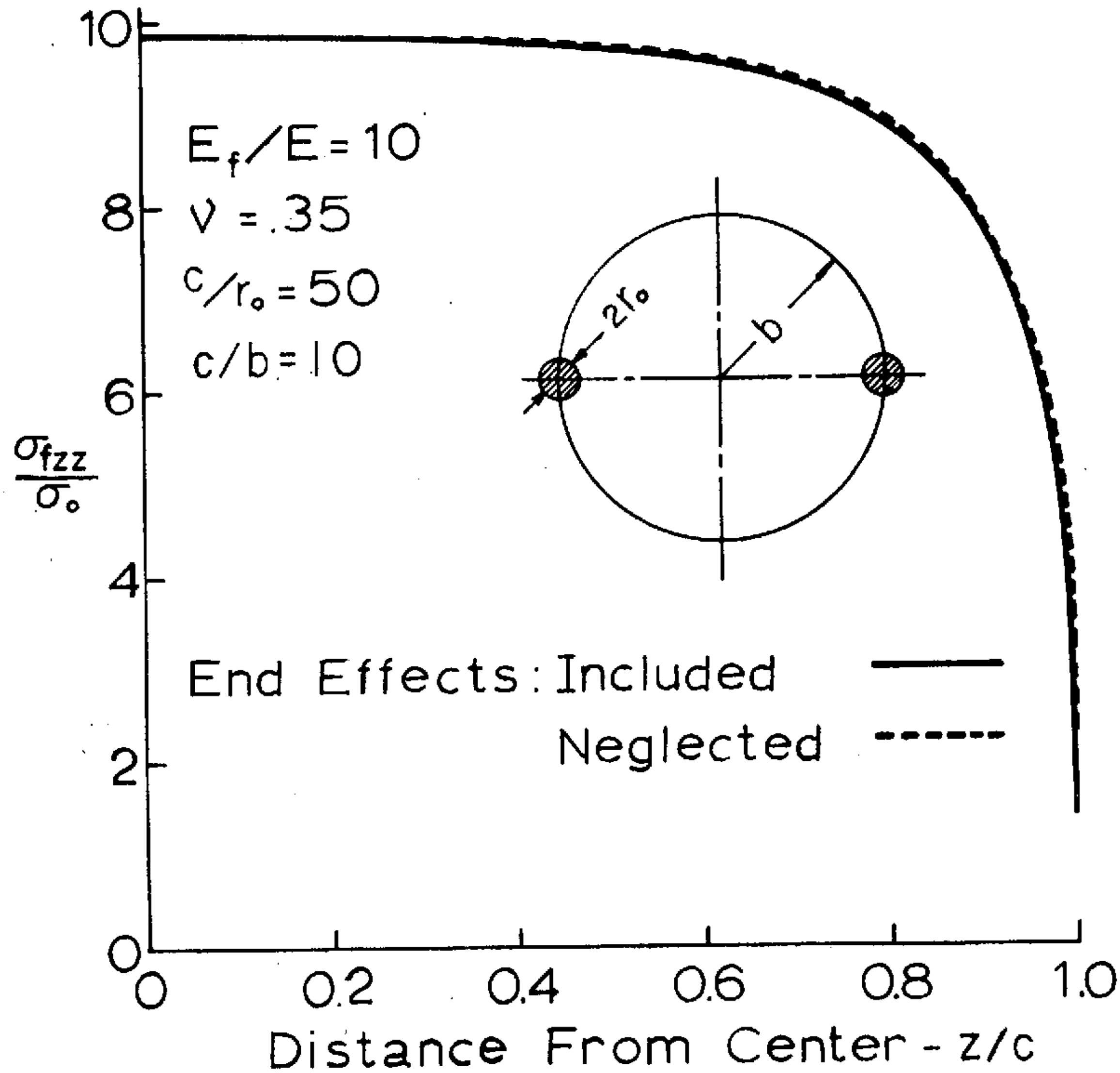


Figure 12. The axial stress in two identical filaments obtained with and without including the effect of the end tractions p.

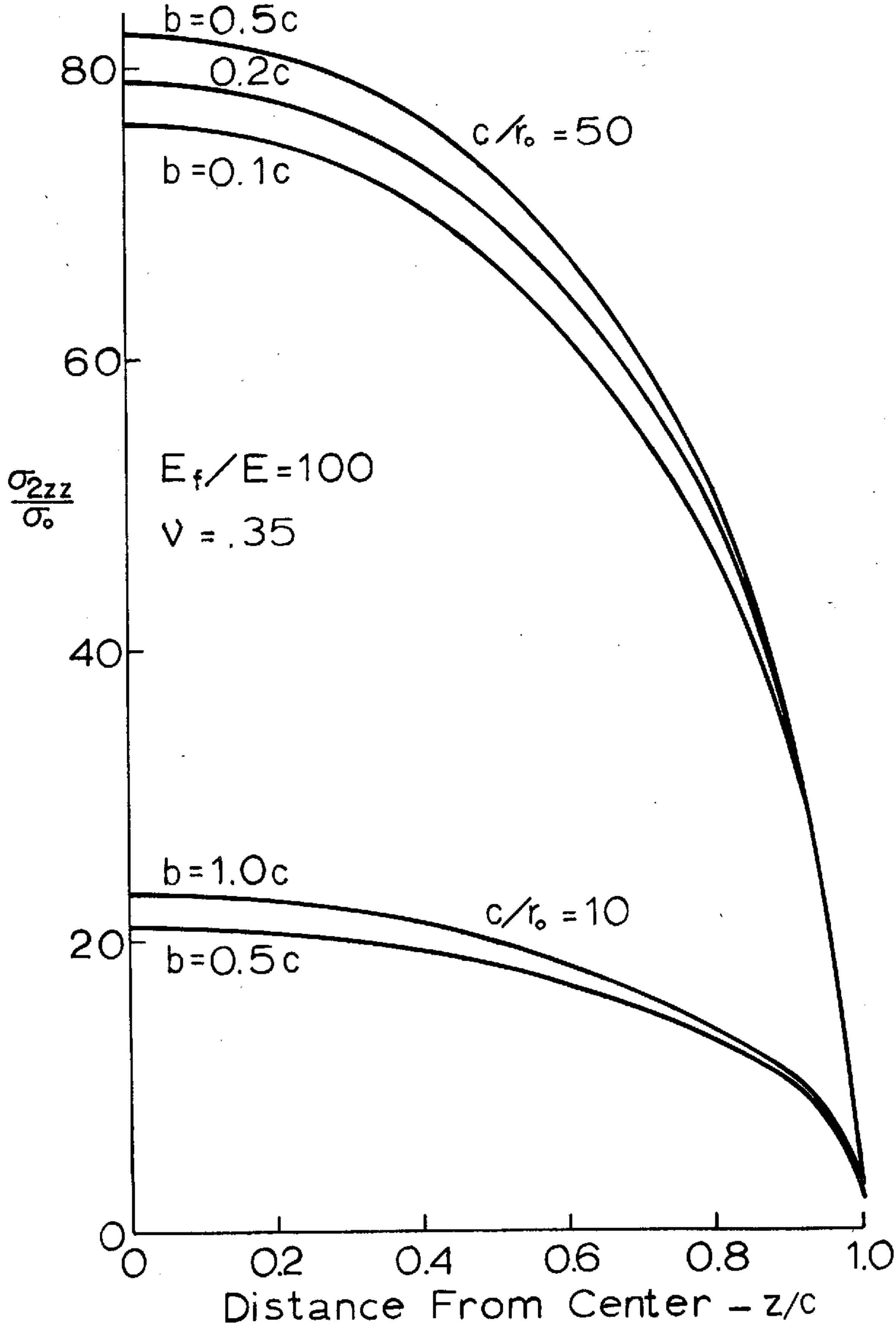


Figure 13. The effect of the length-to-diameter ratio of and the distance between two identical filaments on the axial filament stress.

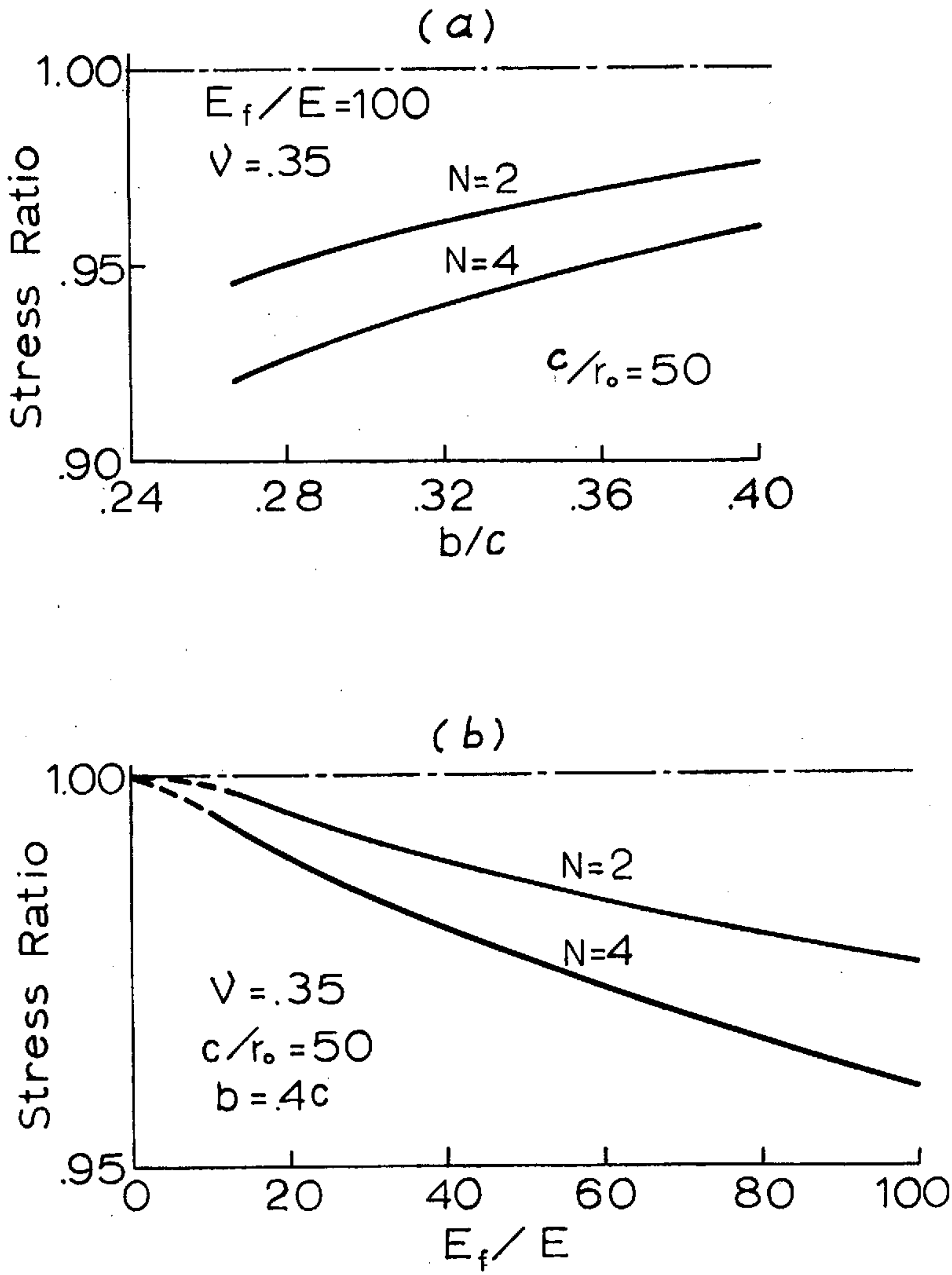


Figure 14. The ratio of the maximum filament stress for $N > 1$ to that for $N = 1$ as a function of the distance parameter b and the modulus ratio E_f/E.

7. Erdogan, F., "Approximate solutions of systems of singular integral equations", SIAM J. Appl. Math., Vol. 17, pp. 1041-1059 (1969).

8. Erdogan, F. and Gupta, G.D., "On the numerical solution of singular integral equations", Q. Appl. Math., Vol. 29, pp. 525-534 (1972).

9. Pacella, A.H., "A penny-shaped crack in a filament reinforced matrix", Ph.D. dissertation, Lehigh University (1970).

APPENDIX A

Integrals used in the derivation of the kernels K_{ij}, $(i,j=1,2)$ given by (32):

$$\rho^2 = 2r_o^2(1-\cos\alpha) + (t-z)^2 , \tag{A.1}$$

$$\int_0^{2\pi}\frac{d\alpha}{\rho} = \frac{4K(k)}{[4r_o^2+(t-z)^2]^{1/2}} , \qquad k^2 = \frac{4r_o^2}{4r_o^2+(t-z)^2} , \tag{A.2}$$

$$\int_0^{2\pi}\frac{\cos\alpha d\alpha}{\rho} = \frac{4K(k)}{[4r_o^2+(t-z)^2]^{1/2}}\left(1+\frac{(t-z)^2}{2r_o^2}\right) - \frac{2}{r_o^2}[4r_o^2+(t-z)^2]^{1/2}E(k) , \tag{A.3}$$

$$\int_0^{2\pi}\frac{d\alpha}{\rho^3} = \frac{4E(k)}{(t-z)^2[4r_o^2+(t-z)^2]^{1/2}} , \tag{A.4}$$

$$\int_0^{2\pi}\frac{(1-\cos\alpha)d\alpha}{\rho^3} = \frac{2}{r_o^2[4r_o^2+(t-z)^2]^{1/2}}[K(k)-E(k)] , \tag{A.5}$$

$$\int_0^{2\pi}\frac{(1-\cos\alpha)^2 d\alpha}{\rho^3} = \frac{2[2r_o^2+(t-z)^2]}{r_o^2[4r_o^2+(t-z)^2]^{1/2}}E(k) - \frac{2(t-z)^2}{r_o^2[4r_o^2+(t-z)^2]^{1/2}}K(k) . \tag{A.6}$$

APPENDIX B

Simplified expressions for the functions $M_i(z)$ and $N_i(z)$ given by (45):

Note that

$$\frac{dN_i(z)}{dz} = (2i-1)(c-z)N_{i+1}(z) \ ,$$

$$\frac{dM_i(z)}{dz} = -(2i-1)(c+z)M_{i+1}(z) \ . \tag{B.1a,b}$$

Also

$$\int_0^{2\pi} \frac{d\theta}{[r^2 + r_o^2 - 2rr_o\cos\theta + (c\pm z)^2]^{1/2}} = \frac{4K(k_{\substack{1\\2}})}{[(r+r_o)^2 + (c\pm z)^2]^{1/2}} , \tag{B.2}$$

$$k_{\substack{1\\2}}^2 = \frac{4rr_o}{(r_o+r)^2 + (c\pm z)^2} ,$$

where the upper and lower indexes in the modulus k correspond to the upper and lower signs in $(c\pm z)$. Thus,

$$M_1(z) = 4\int_0^{r_o} \frac{rK(k_1)dr}{[(r+r_o)^2 + (c+z)^2]^{1/2}} ,$$

$$N_1(z) = 4\int_0^{r_o} \frac{rK(k_2)dr}{[(r+r_o)^2 + (c-z)^2]^{1/2}} , \tag{B.3a,b}$$

and

$$\left.\begin{matrix} M_2(z) \\ N_2(z) \end{matrix}\right\} = 4\int_0^{r_o} \frac{rE(k_{\substack{1\\2}})dr}{[(r+r_o)^2 + (c\pm z)^2]^{1/2}[(r_o-r)^2 + (c\pm z)^2]}$$

$$\left.\begin{matrix} M_3(z) \\ N_3(z) \end{matrix}\right\} = \frac{4}{3}\int_0^{r_o} \frac{rdr}{[(r_o+r)^2 + (c\pm z)^2]^{1/2}[(r_o-r)^2 + (c\pm z)^2]}$$

$$\times \left\{\frac{-K(k_{\substack{1\\2}})}{(r_o+r)^2 + (c\pm z)^2} + 2E(k_{\substack{1\\2}})\left[\frac{1}{(r_o+r)^2 + (c\pm z)^2}\right.\right.$$

$$\left.\left. + \frac{1}{(r_o-r)^2 + (c\pm z)^2}\right]\right\} . \tag{B.4a,b}$$